Engineering Notes

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Efficacy of the Gibbs-Appell Method for Generating Equations of Motion for Complex Systems

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I. Introduction

THERE are many methods for obtaining equations of motion for complex systems. Different methods usually lead to different though equivalent sets of equations and require different amounts of labor in their implementation. Hence, when faced with the necessity of obtaining equations of motion for a complex system, the choice of a method is a critical step in the procedure.

Kane and Levinson[1] have compared seven different methods by applying each to a system consisting of a particle mechanically coupled in a particular manner to a rigid body and have concluded that the best method for handling this system and, by projection, other complex systems is a method that they call Kane's method.[2]

One of the methods considered by Kane and Levinson is the Gibbs-Appell method. In comparing this method with Kane's method, they concede that both methods are equally systematic and lead to the same equations of motion when the same coordinates are used, but they contend that Kane's method is nevertheless clearly superior because it requires less labor to obtain both the inertial terms and the force terms in the equations of motion. We have previously argued[3,4] that both methods yield the same results because Kane's equations are simply a particular form of the Gibbs-Appell equations and Kane's method is simply a particular method of applying the Gibbs-Appell equations. In this Note, we address the question of whether the labor involved using Kane's method to obtain the equations of motion for the foregoing system is truly less than the labor involved using the Gibbs-Appell method.†

We consider a system consisting of a particle mechanically coupled in an *arbitrary* manner to a rigid body, a system that is slightly more general than the one considered by Kane and Levinson, but that contains their system as a special case. Applying the Gibbs-Appell method to this system and fully and efficiently utilizing the fact, inherent in this method, that the dynamical properties of a system of particles or a system of forces can be incorporated in and generated from a single function, we succeed in reducing the labor involved in obtaining both the inertial terms and the force terms in the equations of motion well below the level required using Kane's method. Hence, the conclusion made in Ref. 1 that Kane's method is superior to the Gibbs-Appell method is not only inconsistent with our results, but the opposite conclusion is more defensible.

Received Feb. 27, 1987; revision received Sept. 24, 1987.

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†To be consistent with our previously stated position, we should refer to the first method as Kane's method of applying the Gibbs-Appell equations of motion, and the second method as the standard method of applying the Gibbs-Appell equations of motion. However, to economize on terminology, we refer to the first method as simply Kane's method and the second method as simply the Gibbs-Appell method.

II. Gibbs-Appell Equations of Motion

A simple introduction to the Gibbs-Appell method and its relationship to Kane's method has been given in Ref. 3. In this section, we briefly review the Gibbs-Appell method in order to introduce a few additional concepts and a little known but meritorious variation of the method.

If the configuration of a system of N particles is represented by a single point in a $3N$-dimensional Cartesian configuration space (see, for example, Vol. 1, Chap. 31 of Ref. 5) and if we let f' be the $3N$-dimensional resultant of the given forces acting on the system and f'' the $3N$-dimensional resultant of the constraint forces, then Newton's equations of motion for the system can be written

$$m_i\ddot{x}_i = f_i' + f_i'' \qquad i = 1,\ldots,3N \tag{1}$$

If 1) we let $q \equiv q_1, q_2,\ldots,q_n$ represent any independent set of configuration coordinates that together with the geometric constraints acting on the system uniquely determine the configuration of the system; 2) we let $\dot{r} \equiv \dot{r}_1, \dot{r}_2,\ldots,\dot{r}_{n-m}$ represent any independent set of motional coordinates, linear in the $\dot{q}_i$, that together with the kinematic constraints acting on the system uniquely determine, for a given configuration, the motion of the system; 3) the constraint force f'' satisfies the condition

$$\sum_{j=1}^{3N} f_j'' \frac{\partial \dot{x}_j(q,\dot{r},t)}{\partial \dot{r}_i} = 0 \qquad i = 1,\ldots,n-m \tag{2}$$

and 4) we define

$$S \equiv \frac{1}{2}\sum_{i=1}^{3N} m_i\ddot{x}_i^2 \tag{3}$$

$$U \equiv \sum_{i=1}^{3N} f_i\ddot{x}_i \tag{4}$$

$$R \equiv S - U \tag{5}$$

then the equations of motion for the system can be written

$$\frac{\partial R(q,\dot{r},\ddot{r},t)}{\partial \ddot{r}_i} = 0 \qquad i = 1,\ldots,n-m \tag{6}$$

Additive terms in S, U, or R that do not contain accelerations will not affect the equations of motion. Hence, two values of S, U, or R will be assumed to be equivalent if their difference is independent of any accelerations.

The functions U and R were introduced by Appell in one of his earlier papers,[6] but are generally ignored in most treatments of the Gibbs-Appell method. We have found them quite helpful.

III. The System

We consider a system consisting of a particle P of mass m coupled to a rigid body B of mass M and principal moments of inertia I_1, I_2, and I_3 for the center of mass C. The set of forces other than constraint forces acting on the system are assumed to be equivalent to the following set of forces: an external force F acting at the center of mass C of the rigid body; an external couple T acting on the rigid body; an external force f acting on the particle; an internal action-reaction pair of forces $(\sigma, -\sigma)$ mediated by the coupling mechanism and directed along a line joining the particle and rigid body, with $\boldsymbol{\sigma}$ being the force acting on the particle and $-\boldsymbol{\sigma}$ the force acting on the rigid body.

IV. Notation

Let K_O be a set of space-fixed axes with origin at a point O; K_B a set of axes fixed in the rigid body with origin at the center of mass C of the rigid body and 1, 2, and 3 axes in the principal directions; $\boldsymbol{X}$, $\boldsymbol{V}$, and $\boldsymbol{A}$ the position, inertial velocity, and inertial acceleration, respectively, of C relative to O; $\boldsymbol{x}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ the position, inertial velocity, and inertial acceleration, respectively, of P relative to C; $\boldsymbol{\omega}$ the angular velocity of frame K_B with respect to frame K_O; q_a and q_b any two coordinates that together with the constraints determine the position of P with respect to C; $\dot{r}_a$ and $\dot{r}_b$ any pair of motional coordinates linear in $\dot{q}_a$ and $\dot{q}_b$ that together with the constraints uniquely determines the values of $\dot{q}_a$ and $\dot{q}_b$; ϵ_{ijk} the Levi-Civita symbol that has the value $+1$, -1, or 0 depending on whether ijk is an even permutation of the sequence 1, 2, 3, an odd permutation of the sequence, or neither.

Throughout the remainder of this paper, the range of the subscripts i, j, and k will run from 1-3, and we will omit indicating this fact except when we wish to particularly emphasize it.

V. Equations of Motion for the System

The Gibbs-Appell function S for the given system, *with the components of all quantities expressed with respect to the body-fixed axes* K_B, is given by

$$S = \tfrac{1}{2}M\sum_i A_i^2 + \tfrac{1}{2}m\sum_i (A_i + a_i)^2 + \tfrac{1}{2}\sum_i I_i\dot{\omega}_i^2 + \sum_i\sum_j\sum_k \epsilon_{ijk}I_i\omega_i\dot{\omega}_j\omega_k \tag{7}$$

The first term on the right-hand side of Eq. (7) is the contribution from the center of mass motion of the rigid body, the second term is the contribution from the motion of the particle, and the last two terms are the contribution from the rotational motion of the rigid body. The foregoing expressions can be written down immediately from the definition of the Gibbs-Appell function S for a particle and from general expressions for the Gibbs-Appell function for a rigid body.‡

The Gibbs-Appell potential U for the given system, *with the components of all quantities expressed with respect to the body fixed axes* K_B, is given by

$$U = \sum_i F_iA_i + \sum_i f_i(A_i + a_i) + \sum_i T_i\dot{\omega}_i + \sum_i \sigma_i\ddot{x}_i \tag{8}$$

The first two terms on the right-hand side of Eq. (8) are the contribution from the forces $\boldsymbol{F}$ and $\boldsymbol{f}$, respectively, and follow immediately from the definition of U. The third and fourth terms are the contributions from the torque $\boldsymbol{T}$ and the action-reaction pair $(\boldsymbol{\sigma}, -\boldsymbol{\sigma})$. Their derivation, which we leave to the reader, requires a little care but can be accomplished without undue effort.

The equations generated using the aforementioned expressions in the Gibbs-Appell equations of motion will depend on our choice of motional coordinates. To permit comparison of our results with the results of Kane we choose the following motional coordinates: V_1, V_2, V_3, ω_1, ω_2, ω_3, $\dot{r}_a$, $\dot{r}_b$; where the components V_i and ω_i are the components of $\boldsymbol{V}$ and $\boldsymbol{\omega}$ *with respect to the body-fixed axes* K_B, and $\dot{r}_a$ and $\dot{r}_b$ have been defined in the preceding section.

The quantities S and U can be written in terms of the foregoing motional coordinates by noting from the relations connecting the time rate of change of a vector quantity as noted by an observer in the inertial frame K_O with the time rate of change of the same vector quantity as noted by observer in the body frame K_B that

$$A_i = \dot{V}_i + \sum_j\sum_k \epsilon_{ijk}\omega_jV_k \tag{9}$$

$$a_i = \ddot{x}_i + \sum_j\sum_k \epsilon_{ijk}\dot{\omega}_jx_k + 2\sum_j\sum_k \epsilon_{ijk}\omega_j\dot{x}_k - x_i\sum_j \omega_j^2 + \omega_i\sum_j \omega_jx_j \tag{10}$$

and from the definitions of q_a, q_b, $\dot{r}_a$, and $\dot{r}_b$ that

$$x_i = x_i(q_a, q_b, t) \tag{11}$$

$$\dot{x}_i = \dot{x}_i(q_a, q_b, \dot{r}_a, \dot{r}_b, t) \tag{12}$$

$$\ddot{x}_i = x_i(q_a, q_b, \dot{r}_a, \dot{r}_b, \ddot{r}_a, \ddot{r}_b, t) \tag{13}$$

With the foregoing choice of motional coordinates, the Gibbs-Appell equations of motions of motion are

$$\partial R/\partial\dot{V}_i = 0 \qquad i = 1, 2, 3 \tag{14}$$

$$\partial R/\partial\dot{\omega}_i = 0 \qquad i = 1, 2, 3 \tag{15}$$

$$\partial R/\partial\ddot{r}_\alpha = 0 \qquad \alpha = a, b \tag{16}$$

where $R \equiv S - U$. Substituting Eqs. (7) and (8) in Eqs. (14-16) and making use of Eqs. (9-13) to carry out the derivatives of A_i and a_i, we immediately obtain

$$MA_i + m(A_i + a_i) = F_i + f_i \qquad i = 1, 2, 3 \tag{17}$$

$$m\sum_j\sum_k \epsilon_{ijk}x_j(A_k + a_k) + I_i\dot{\omega}_i - \sum_j\sum_k \epsilon_{ijk}I_j\omega_j\omega_k = T_i + \sum_j\sum_k \epsilon_{ijk}x_jf_k \qquad i = 1, 2, 3 \tag{18}$$

$$m\sum_i (A_i + a_i)(\partial\ddot{x}_i/\partial\ddot{r}_\alpha) = \sum_i (f_i + \sigma_i)(\partial\ddot{x}_i/\partial\ddot{r}_\alpha) \qquad \alpha = a, b \tag{19}$$

Equations (17-19), with A_i and a_i given by Eqs. (9) and (10), respectively, are the desired results. They could also have been obtained quite simply, but not nearly as neatly, by judiciously combining Newton's equations of motion for the particle, Newton's equations of motion for the center of mass of the rigid body, Euler's equations, and the general equations of motion for a particle in an accelerated frame, specifically the frame K_B.

The system of Kane and Levinson is a special case of the general system described here. The equations of motion that they obtain can be easily obtained by identifying in Eqs. (17-19) the values of q_α, $\dot{r}_\alpha$, x_i, and σ_i corresponding to their choice of mechanical linkage and coordinates. Specifically, if in Eqs. (17-19) we let $q_a = r$, $q_b = \theta$, $\dot{r}_a = b\dot{\theta}$, $\dot{r}_b = b\dot{\theta}\sin\theta - \dot{r}$, $x_1 = b\sin\theta$, $x_2 = c - b\cos\theta - r$, $x_3 = 0$, $\sigma_1 = (2\tau/b\cos\theta) - \sigma\tan\theta$, and $\sigma_2 = \sigma$; and next note that $\partial x_1/\partial r_a = \cos\theta$, $\partial x_2/\partial r_a = 0$, $\partial x_3/\partial r_a = 0$, $\partial x_1/\partial r_b = 0$, $\partial x_2/\partial r_b = 1$, and $\partial x_3/\partial r_b = 0$, then we immediately obtain Eqs. (63-70) in Ref. 1.

VI. Conclusion

To prove the superiority of one method over another on the basis of one example is an extremely difficult task, especially when one is an advocate of a particular method. Hence, we make no absolute claim for the general superiority of the Gibbs-Appell method for obtaining equations of motion for complex systems. However, the results of this Note do demonstrate that the Gibbs-Appell method should be seriously considered by anyone faced with the need to determine and analyze the motion of a complex system.

‡See, for example, Vol. 2, Chap. 70 of Ref. 5. In using this reference, note that there is an error in Theorem 2. The stated result is valid only if the frame R is an inertial frame or a body frame. An additional term occurs for other frames. For the frame used in this paper, the theorem is correct.

References

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Pitch Pointing Flight Control System Design in the Frequency Domain

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Introduction

ADVANCED aircraft provide some significant modes for bombing and air-to-air combat. One of the main control objectives for these advanced modes is the decoupling of the multivariable aircraft system. For the longitudinal dynamics of an aircraft, one such decoupled mode is pitch pointing, which is characterized by decoupling the pitch attitude and flight-path angle.

Sobel and Shapiro[1] have proposed a design methodology that uses eigenstructure assignment to decouple the system, and have obtained the desired properties of the closed-loop feedback system in state-space. In this Note, an alternative frequency domain design that decouples the multivariable system by using an H^∞-optimization technique is proposed, and a stable minimum-phase weighting function to meet the desired damping ratio and the rise time is constructed. Using the H^∞-optimization technique, the multivariable problem is treated exactly as the scalar problem in the pitch pointing control design, and a diagonal closed-loop transfer function matrix of the multivariable system is obtained. The result of the pitch pointing controller design shows the perfect system decoupling, which achieves an attitude command tracking but without causing any flight path change, and possesses desired tracking properties.

H^∞-Optimization Design Methodology

Consider the dynamical equation of the aircraft plant in the pitch pointing control system, which is described in Fig. 1 by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Dx \tag{2}$$

The plant is modeled by the loop transfer function matrix

$$P = D(sI - A)^{-1} B \tag{3}$$

Received April 17, 1987; revision received Oct. 6, 1987.

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If $\det(sI - A)$ has right-half-plane (RHP) zeros, then this kind of aircraft without augmentation is unstable. For such an aircraft, does there exist a controller $C(s)$ such that the closed-loop multivariable system is decoupled, and possesses an optimal asymptotic tracking property under some design criterion? In this Note, we propose an H^∞-optimization design criterion as follows:

Theorem: Consider the $n \times n$ multivariable system given in Fig. 1. Let the input r be a unit step function, T_o be the closed-loop transfer function matrix, and W be a stable minimum-phase diagonal weighting function matrix. If $\det(sI - A)$ in Eq. (3) has RHP zeros, then there exists a diagonal optimal all-pass function matrix $\tilde{G}_o \in H^\infty$, the bounded and analytic function matrix on RHP, such that

$$\|\tilde{G}_o(s)\|_\infty = \inf_{C(s)\in H^\infty} \|W(s)T_o(s)\|_\infty = \tilde{k} \tag{4}$$

and the overall transfer function matrix in steady state would be

$$T(0) = K\tilde{T}_o(0) = KW^{-1}(0)\tilde{G}_o(0) = I \tag{5}$$

where W is viewed as a design parameter selected to reflect the desired properties of the optimal transfer function matrix $\tilde{T}_o$ and make the infimum $\tilde{k}$ less than one. (The sufficient condition[3] $\inf_{C\in H^\infty} \|WT_o\|_\infty < 1$ assures the closed-loop system optimal robust for all multiplicative plant perturbation $\Delta_m(s)$ with $\bar{\sigma}[\Delta_m(j\omega)] < \bar{\sigma}[W(j\omega)] \forall \omega \geq 0$, where $\bar{\sigma}[\cdot]$ denotes the maximum singular value of the matrix $[\cdot]$.) K is a constant-gain matrix chosen to make the overall system asymptotic tracking; $\| \; \|_\infty$ is the maximum modulus norm, and $\tilde{T}_o$ denotes the optimal closed-loop transfer function matrix.

This theorem implies that if the diagonal all-pass function matrix $\tilde{G}_o$ is attained, then the closed-loop transfer function matrix $\tilde{T}_o(s) = W^{-1}(s)\tilde{G}_o(s)$ can be decoupled, and the optimal asymptotic tracking property as Eq. (5) will be obtained.

Proof: Let $P = Nd^{-1}$, where d is the Blaschke product of the RHP zeros of $\det(sI - A)$. Select X_i, $Y_i \in H^\infty$, $i = 1,2$, such that

$$X_1 N + dY_1 = I, \quad NX_2 + dY_2 = I \text{ (Bezout Identity)} \tag{6}$$

Then T_o can be parametrized as[3]

$$T_o = I - dY_2 + dNR \tag{7}$$

and the problem in Eq. (4) can be reduced to find

$$\inf_{C\in H^\infty} \|WT_o\|_\infty \tag{8}$$

$$= \inf_{R\in H^\infty} \|W(NX_2 + dNR)\|_\infty \quad \text{[by Eq. (6) and (7)]} \tag{9}$$

$$= \inf_{R\in H^\infty} \|W(N)_i(N)_o(X_2 + dR)\|_\infty$$

[by inner-outer factorization $N = (N)_i(N)_o$] (10)

$$= \inf_{R'\in H^\infty} \|W(N)_o X_2 + dR'\|_\infty$$

[assume that $(N)_i$ is diagonal in the following

example $(N)_i = I$] (11)

$$= \inf_{R'\in H^\infty} \|F + dR'\|_\infty \tag{12}$$

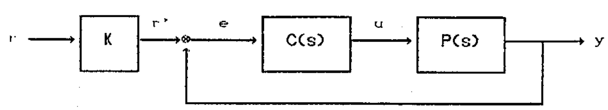

Fig. 1 Typical control system.